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Why pre-tensioning stiffens cable systems

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Abstract

It is proved that the potential energy of any pin-bar assembly with totally tensioned members possesses strict minimum independently of the assembly topology, geometry and magnitudes of member forces. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

The fact that pre-tensioning stiffens cable systems is well-known to engineers and widely used in design. Probably, the experience of stiffening a single string by pre-tension served as a spur for the development of cable systems. The phenomenon of stiffening strings by pretension is observed by almost everybody, so it does not need any theoretical support and may be accepted as a law of nature. However, the situation with cable systems is much more subtle. Only few of the infinite number of possible pre-tensioned cable systems were practically realized. Though stiffening has been observed in all these cases, there is not enough evidence to extrapolate from these results to other systems that have not been examined yet.

A more accurate and general formulation of the problem may be proceeded in the following way. An assembly of straight members resisting to tension or compression and jointed by pins is considered. This assembly is, generally, a space truss or, in the particular case of total tensioning, it may be a cable system. Stiffness of a specific state of the assembly is defined as a stability of this state from the point of view of the minimum of the potential energy corresponding to it. The latter means that the *tangent stiffness matrix* of the assembly corresponding to the examined state must be strictly positive definite. To summarize: positive definiteness of the tangent stiffness matrix means that the assembly possesses stiffness at the considered state. The observed experimental facts of stiffening cable systems by pretension suggest the following general question: *Does total tensioning provide stability of an assembly independently of its topology, geometry and specific magnitudes of member forces*? The answer is: *yes, it does*. The proof suggested below is

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based on the analysis of location of eigenvalues of the tangent stiffness matrix which, in turn, is based on the extension of some results of linear algebra on location of matrix eigenvalues.

Though the contents of the second section of the present note is motivated by the contents of the fourth section, we found it more suitable to change the 'historical' order for the sake of compactness of the presentation.

2. An extension of some linear algebra results on location of eigenvalues

Let $G(\mathbf{A})$ designates the union of r Gershgorin discs where all eigenvalues of an r by r matrix $\mathbf{A}$ are located:

$$G(\mathbf{A}) \equiv \bigcup_{i=1}^{r} \left\{ z : |z - a_{ii}| \leqslant R_i'(\mathbf{A}) \right\}, \quad \text{where } R_i'(\mathbf{A}) = \sum_{\substack{j=1 \\ j \neq i}}^{r} |a_{ij}|, \quad 1 \leqslant i \leqslant r.$$

Then the following lemma takes place.

Lemma. Let $\mathbf{A}$ be an r by r matrix and let λ be an eigenvalue of $\mathbf{A}$ that lies on the boundary of $G(\mathbf{A})$. Let $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$, and suppose p is an index such that

$$|x_p| = \max_{1 \leqslant i \leqslant r} |x_i| = \|x\|_\infty \neq 0.$$

Then

1. If k is any index such that $|x_k| = |x_p|$, then $|\lambda - a_{kk}| = R_k'$; that is, the kth Gershgorin circle passes through λ; and
2. If $|x_k| = |x_p|$ for some $k = 1, \ldots, r$ and if $a_{kj} \neq 0$ for some $j \neq k$, then $|x_j| = |x_p|$ as well.

Proof. See Horn and Johnson (1985).

Let *property GSC* be introduced now.

Definition 1. An r by r matrix $\mathbf{A}$ is said to have *the generalized property SC (GSC)* if it is possible to find g, $1 \leqslant g \leqslant r$, nonintersecting sets of distinct integers (covering all integers from 1 to r) among integers from 1 to r, so that for every pair of integers p_t, q_t of the tth set there is a sequence of distinct integers, belonging to the same set, $k_1 = p_t, k_2, \ldots, k_m = q_t$, $1 \leqslant m \leqslant r$, such that all of the matrix entries $a_{k_1k_2}, a_{k_2k_3}, \ldots, a_{k_{m-1}k_m}$ are nonzero.
Then the following theorem and its corollary may be formulated and proved.

Theorem. Let $\mathbf{A}$ be an r by r matrix, and suppose that λ is an eigenvalue of $\mathbf{A}$ that lies on the boundary of $G(\mathbf{A})$. If $\mathbf{A}$ has *property GSC*, then

1. Every Gershgorin circle of at least one of g sets, defined by property GSC, passes through λ; and
2. If $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$, then $|x_i| = |x_j|$ for at least all i, j belonging to the same set.

Proof. Let $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ with $|x_i| \leqslant |x_{p_t}| = \|x\|_\infty$ for all $I = 1, \ldots, r$. Then $|\lambda - a_{p_tp_t}| = R_{p_t}'$ by the Lemma. Let q_t be any other index which belongs to the same tth set as p_t, $1 \leqslant q_t \leqslant r$, $q_t \neq p_t$. Since $\mathbf{A}$ has property GSC, there is a sequence of distinct indices $k_1 = p_t, k_2, \ldots, k_m = q_t$ such that all of the matrix entries $a_{k_1k_2}, a_{k_2k_3}, \ldots, a_{k_{m-1}k_m}$ are nonzero. Since $a_{k_1k_2} = a_{p_tk_2} \neq 0$, we conclude by assertion (2) of the Lemma that $|x_{p_t}| = |x_{k_2}|$. But then $a_{k_2k_3} \neq 0$ and so $|x_{k_2}| = |x_{k_3}| = |x_{p_t}|$. Proceeding in this way we obtain that $|x_{k_i}| = |x_{p_t}|$ for all $i = 1, \dots, m$ and hence, by assertion (1) of the Lemma, $|\lambda - a_{k_mk_m}| = |\lambda - a_{q_tq_t}| = R'_{q_t}$; that is, the qth Gershgorin circle passes through λ and $|x_{p_t}| = |x_{q_t}|$. But since q_t was an arbitrary index among those belonging to the tth set, we conclude that every Gershgorin circle of this set passes through λ and that $|x_i| = |x_{p_t}|$ for at least all i belonging to the set. (*End of the proof*).

Corollary. Let $\mathbf{A}$ be an r by r matrix and suppose that $\mathbf{A}$ has property *GSC*. If $\mathbf{A}$ is diagonally dominant, that is $|a_{ii}| \geqslant R'_i$, and $|a_{jj}| > R'_j$ for at least one value of $j = k_1 = p_t, k_2, \dots, k_m = q_t$ for every of g sets, the $\mathbf{A}$ is invertible.

Proof. If $\mathbf{A}$ were not invertible, then 0 would be an eigenvalue of $\mathbf{A}$. Since $\mathbf{A}$ is diagonally dominant, 0 cannot be an interior point of $G(\mathbf{A})$ and hence it must be a boundary point. The theorem says that every Gershgorin circle of at least one of g sets defined by property *GSC* must pass through 0, but there is $|a_{jj}| > R'_j$ for every of g sets, then the jth circle cannot pass through 0. Consequently, no one of Gershgorin circles passes through 0. (*End of the proof*).

By assuming $g = 1$ in the above stated Definition 1 of property *GSC*, we obtain the well-known property *SC* and the Theorem and its Corollary are reduced to the 'better theorem' and 'better corollary' presented by Horn and Johnson (1985).

Property *GSC* of matrix $\mathbf{A}$ may be visualized by introducing a concept of a *weakly connected graph*.

Definition 2. A directed graph Γ consisting of r nodes is said to be *weakly connected* if there are g with $1 \leqslant g \leqslant r$ nonintersecting sets of distinct nodes (covering all nodes from 1 to r) among nodes from 1 to r, so that for every pair of nodes p_t, q_t of the tth set there is a directed path of finite length, belonging to the same set, that begins at p_t and ends at q_t.

The property *GSC* of matrix $\mathbf{A}$ means that graph $\Gamma(\mathbf{A})$ of this matrix is weakly connected and vice versa. Again, assuming $g = 1$ in Definition 2 the 'weakly connected' should be replaced by the 'strongly connected' as in Horn and Johnson (1985).

3. The structure of the tangent stiffness matrix

The internal strain energy of an arbitrary pin-bar assembly takes the following form:

$$V = \sum_{i=1}^{m} \left(\frac{1}{2} \frac{E_i A_i}{l_i} \Delta_i^2 + P_i \Delta_i \right) \tag{1}$$

where $E_i, A_i, l_i, P_i, \Delta_i$ are the ith member elasticity modulus, cross-section area, initial length, initial force and elongation correspondingly. Member elongations and, consequently, the strain energy is a function of nodal displacements U_i.

In the case of the 'dead' load the current state of the assembly is stable where the *Hessian* or *tangent stiffness matrix* $\partial^2 V/\partial U_i \partial U_j$ is positive definite at the vicinity of the current state: $\mathbf{U} = \mathbf{0}$. The zero means that the current state is referred to as the initial one for the sake of simplicity. This criterion is general, but it may be simplified in the case of *kinematically indeterminate* assemblies (Calladine and Pellegrino, 1991; Kuznetsov, 1991; Volokh and Vilnay, 1997).

In principle, analysis of the initial tangent stiffness matrix may be carried out numerically for every particular case. However, in this work the properties of the initial tangent stiffness matrix are investigated without referring to a specific structural topology, geometry or member forces' distribution.

An arbitrary kth row of the stiffness matrix takes the form:

$$a_{ki} = \frac{\partial}{\partial U_i}\left(\frac{\partial V}{\partial U_k}\right)\Big|_{\mathrm{U}=0} \quad i = 1, \ldots r \tag{2}$$

where r is the number of degrees of freedom.

Taking into account that U_k is included only into the change of the length of the members corresponding to the appropriate node, it is possible to reduce the strain energy expression given in eqn (2) to the form:

$$V_1 = \sum_{n=1}^{\rho}\left\{\frac{E_{i_n}A_{i_n}}{2l_{i_n}}\Delta_{i_n}^2 + P_{i_n}\Delta_{i_n}\right\}$$

$$\Delta_{i_n} = \tilde{l}_{i_n} - l_{i_n}$$

$$\tilde{l}_{i_n} = \sqrt{\left(X_k + U_k - X_{s_n} - U_{s_n}\right)^2 + \left(X_{k+1} + U_{k+1} - X_{s_n+1} - U_{s_n+1}\right)^2 + \left(X_{k+2} + U_{k+2} - X_{s_n+2} - U_{s_n+2}\right)^2};$$

$$l_{i_n} = \sqrt{\left(X_k - X_{s_n}\right)^2 + \left(X_{k+1} - X_{s_n+1}\right)^2 + \left(X_{k+2} - X_{s_n+2}\right)^2} \tag{3}$$

(see also Fig. 1).

In this case the first and the second derivatives take the form:

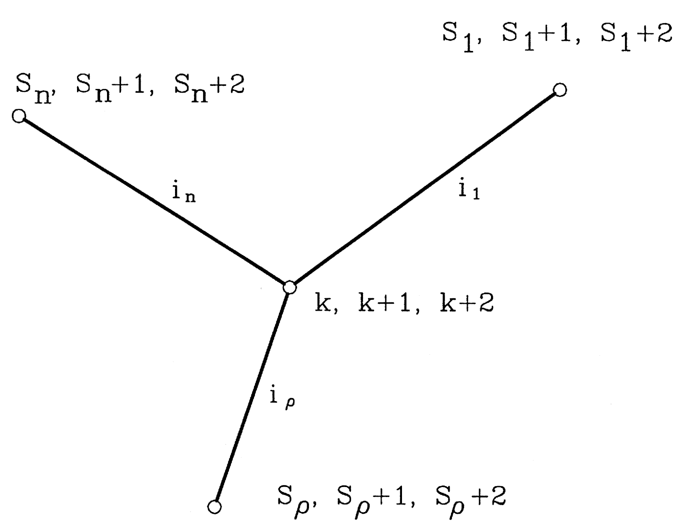

Fig. 1. An arbitrary node of the assembly.

$$\frac{\partial V}{\partial U_k} = \frac{\partial V_1}{\partial U_k} = \sum_{n=1}^{\rho} \left\{ \frac{E_{i_n} A_{i_n}}{l_{i_n}} \Delta_{i_n} \frac{\partial \Delta_{i_n}}{\partial U_k} + P_{i_n} \frac{\partial \Delta_{i_n}}{\partial U_k} \right\} \tag{4}$$

$$\frac{\partial V}{\partial U_i \partial U_k} = \frac{\partial V_1}{\partial U_i \partial U_k} = \sum_{n=1}^{\rho} \left\{ \frac{E_{i_n} A_{i_n}}{l_{i_n}} \left(\frac{\partial \Delta_{i_n}}{\partial U_i} \frac{\partial \Delta_{i_n}}{\partial U_k} + \Delta_{i_n} \frac{\partial^2 \Delta_{i_n}}{\partial U_i \partial U_k} \right) + P_{i_n} \frac{\partial^2 \Delta_{i_n}}{\partial U_i \partial U_k} \right\} \tag{5}$$

It is possible to obtain, omitting details, the following formulae for $i = k, k+1, k+2$ and $i = s_n, s_n + 1, s_n + 2$:

$$\frac{\partial \Delta_{i_n}}{\partial U_k} = -\frac{\partial \Delta_{i_n}}{\partial U_{s_n}} = \frac{X_k + U_k - X_{s_n} - U_{s_n}}{\tilde{l}_{i_n}}$$

$$\frac{\partial \Delta_{i_n}}{\partial U_{k+1}} = -\frac{\partial \Delta_{i_n}}{\partial U_{s_n+1}} = \frac{X_{k+1} + U_{k+1} - X_{s_n+1} - U_{s_n+1}}{\tilde{l}_{i_n}} \tag{6}$$

$$\frac{\partial \Delta_{i_n}}{\partial U_{k+2}} = -\frac{\partial \Delta_{i_n}}{\partial U_{s_n+2}} = \frac{X_{k+2} + U_{k+2} - X_{s_n+2} - U_{s_n+2}}{\tilde{l}_{i_n}}$$

$$\frac{\partial^2 \Delta_{i_n}}{\partial U_k^2} = -\frac{\partial^2 \Delta_{i_n}}{\partial U_{s_n} \partial U_k} = \frac{1}{\tilde{l}_{i_n}} \left(1 - \left(\frac{\partial \tilde{l}_{i_n}}{\partial U_k} \right)^2 \right) = \frac{1}{\tilde{l}_{i_n}} \left(1 - \left(\frac{\partial \Delta_{i_n}}{\partial U_k} \right)^2 \right) \tag{7}$$

$$\frac{\partial^2 \Delta_{i_n}}{\partial U_{k+1} \partial U_k} = -\frac{\partial^2 \Delta_{i_n}}{\partial U_{s_n+1} \partial U_k} = -\frac{1}{\tilde{l}_{i_n}} \frac{\partial \tilde{l}_{i_n}}{\partial U_{k+1}} \frac{\partial \tilde{l}_{i_n}}{\partial U_k} = -\frac{1}{\tilde{l}_{i_n}} \frac{\partial \Delta_{i_n}}{\partial U_{k+1}} \frac{\partial \Delta_{i_n}}{\partial U_k}$$

$$\frac{\partial^2 \Delta_{i_n}}{\partial U_{k+2} \partial U_k} = -\frac{\partial^2 \Delta_{i_n}}{\partial U_{s_n+2} \partial U_k} = -\frac{1}{\tilde{l}_{i_n}} \frac{\partial \tilde{l}_{i_n}}{\partial U_{k+2}} \frac{\partial \tilde{l}_{i_n}}{\partial U_k} = -\frac{1}{\tilde{l}_{i_n}} \frac{\partial \Delta_{i_n}}{\partial U_{k+2}} \frac{\partial \Delta_{i_n}}{\partial U_k} \tag{8}$$

By substituting eqns (6)–(8) into eqn (5) and assuming $\mathbf{U} = \mathbf{0}$, the elements of the kth row of the stiffness matrix take the form

$$a_{ks_n} = -\left(\frac{E_{i_n} A_{i_n}}{l_{i_n}} - \frac{P_{i_n}}{l_{i_n}} \right) \left(\frac{X_k - X_{s_n}}{l_{i_n}} \right)^2 - \frac{P_{i_n}}{l_{i_n}} \tag{9}$$

$$a_{ks_n+1} = -\left(\frac{E_{i_n} A_{i_n}}{l_{i_n}} - \frac{P_{i_n}}{l_{i_n}} \right) \left(\frac{X_k - X_{s_n}}{l_{i_n}} \right) \left(\frac{X_{k+1} - X_{s_n+1}}{l_{i_n}} \right)$$

$$a_{ks_n+2} = -\left(\frac{E_{i_n} A_{i_n}}{l_{i_n}} - \frac{P_{i_n}}{l_{i_n}} \right) \left(\frac{X_k - X_{s_n}}{l_{i_n}} \right) \left(\frac{X_{k+2} - X_{s_n+2}}{l_{i_n}} \right) \tag{10}$$

$$a_{kk} = -\sum_{n=1}^{\rho} a_{ks_n} \tag{11}$$

$$a_{kk+1} = -\sum_{n=1}^{\rho} a_{ks_n+1}$$

$$a_{kk+2} = -\sum_{n=1}^{\rho} a_{ks_n+2} \tag{12}$$

It is important to note that in case of a support point without degrees of freedom s_n, s_n+1, s_n+2 the corresponding elements a_{ks_n}, a_{ks_n+1}, a_{ks_n+2} should be zeroed and corresponding matrix rows and columns removed, while elements a_{kk}, a_{kk+1}, a_{kk+2} are calculated without changes in accordance with eqns (9)–(12).

The stiffness matrix $\mathbf{A}$ whose kth row elements are presented by eqns (9)–(12) may be written in the matrix form as follows

$$\mathbf{A} = \mathbf{B}^T\overline{\mathbf{S}}\mathbf{B} + \mathbf{D} \tag{13}$$

An m by r matrix $\mathbf{B}$ is a standard (kinematic) matrix of direction cosines: $b_{i_nk} = (X_k - X_{s_n})/l_{i_n}$.

An m by m matrix $\overline{\mathbf{S}}$ is a *modified uncoupled stiffness matrix* with nonzero diagonal elements: $\overline{\mathbf{S}}_{i_ni_n} = A_{i_n}(E_{i_n} - \sigma^0_{i_n})/l_{i_n}$, where $\sigma^0_{i_n}$ is the current stress in the i_nth member. The standard uncoupled stiffness matrix $\mathbf{S}$ is obtained from $\bar{\mathbf{S}}$ by zeroing the initial stresses.

An r by r symmetric matrix $\mathbf{D}$ is the *geometric stiffness matrix* with elements:

$$d_{ks_n} = -P_{i_n}/l_{i_n}, \quad d_{ks_n+1} = 0, \quad d_{ks_n+2} = 0 \tag{14}$$

$$d_{kk} = -\sum_{n=1}^{\rho} d_{ks_n}, \quad d_{kk+1} = 0, \quad d_{kk+2} = 0 \tag{15}$$

Again non-diagonal element d_{ks_n} should be zeroed and proper rows and columns removed in case of a support point corresponding to the s_nth degree of freedom, while the same term should be kept on the right hand side of eqn (15_1).

4. Positive definiteness of the tangent stiffness matrix

The proof of positive definiteness of matrix $\mathbf{A}$ includes two steps. At the first step we prove that the first term on the right hand side of eqn (13) is positive *semi*-definite, and at the second step we prove that the second term on the right hand side of eqn (13) is strictly positive definite.

4.1. Step 1

Note that elements of diagonal matrix $\overline{\mathbf{S}}$ are positive, that is $E_{i_n} > \sigma^0_{i_n}$. This assumption means that the current stress cannot be larger in magnitude than the elasticity modulus and it seems to be correct for all existing materials. Moreover, our considerations are restricted by the elastic region of the material behavior. Thus introducing diagonal matrix $\sqrt{\mathbf{S}}$ whose elements are square roots of the corresponding elements of matrix $\overline{\mathbf{S}}$ and designating Euclidean norm as $\| \; \|_2$, it is possible to obtain for an arbitrary vector $\mathbf{x}$:

$$\mathbf{x}^T\mathbf{B}^T\,\overline{\mathbf{S}}\mathbf{B}\mathbf{x} = \mathbf{x}^T\mathbf{B}^T\sqrt{\overline{\mathbf{S}}}^T\sqrt{\overline{\mathbf{S}}}\mathbf{B}\mathbf{x} = \|\sqrt{\overline{\mathbf{S}}}\mathbf{B}\mathbf{x}\|_2^2 \geqslant 0 \tag{16}$$

and the first step is completed.

4.2. Step 2

This step is decomposed, in turn, into the following two moves:

1. It immediately follows from eqns (14) and (15) with account for total tension ($P_{i_n} > 0$) that matrix **D** enjoys diagonal dominancy:

$$d_{kk} \geqslant \sum_{n=1}^{\rho} |d_{ks_n}| = \sum_{\substack{i=1\\ i\neq k}}^{r} |d_{ki}| = R_k'(\mathbf{D}) \tag{17}$$

   This weak inequality transforms into equality if the terms corresponding to support points are not present in the kth row, otherwise, strict inequality is obtained. On the other hand, d_{kk} is positive for all k. The latter means that Gershgorin discs G(**D**) do not include negative numbers and matrix **D** is, at least, positive semi-definite.
2. In order to prove that **D** is strictly positive definite, it is necessary to prove that it is invertible. The theory developed in the second section of the paper serves this purpose.

Let us show that the Corollary is applicable to matrix **D**.

The fact that matrix **D** is diagonally dominant has been already proved above. Let us show that matrix **D** enjoys property *GSC*. For this purpose, we distinguish three sets ($g = 3$) of matrix elements:

$$d_{kk},\ d_{ks_n};$$

$$d_{k+1k+1},\ d_{k+1s_n+1};$$

$$d_{k+2k+2},\ d_{k+2s_n+2};$$

where indices of the first set correspond to degrees of freedom in the x direction; indices of the second set correspond to degrees of freedom in the y direction; and indices of the third set correspond to degrees of freedom in the z direction of the global coordinate system. The graph of matrix **D** is decomposed into three non-connected parts in accordance with above defined sets. Every part of the graph is nothing but the physical structure itself. Thus graph Γ(**D**), being strongly connected for every of this three parts (otherwise the pin-bar assembly is incompatible, that is it does not exist), is weakly connected as a whole.

Finally, every of the above mentioned sets possesses, at least, one integer j corresponding to a supporting point, such that

$$d_{jj} > \sum_{\substack{i=1\\ i\neq j}}^{r} |d_{ji}| = R_j'(\mathbf{D}) \tag{18}$$

Summarizing, we conclude that matrix **D** satisfies all requirements of the Corollary and, therefore, it is invertible. That completes the general proof of positive definiteness of **D** and, consequently, of **A**.

5. Conclusion

It was proved that the equilibrium state of any pin-bar assembly is stable under condition that all members are tensioned. This result does not depend on specific structural topology, geometry and magnitudes of member forces and, consequently, explains why pre-tensioning always stiffens cable systems.

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